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Impact of supersymmetry on the nonperturbative dynamics of fuzzy spheres

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Abstract: We study a 4d supersymmetric matrix model with a cubic term, which incorporates fuzzy spheres as classical solutions, using Monte Carlo simulations and perturbative calculations. The fuzzy sphere in the supersymmetric model turns out to be always stable if the large-N limit is taken in such a way that various correlation functions scale. This is in striking contrast to analogous bosonic models, where the fuzzy sphere decays into the pure Yang-Mills vacuum due to quantum effects when the coefficient of the cubic term becomes smaller than a critical value. We also find that the power-law tail of the eigenvalue distribution, which exists in the supersymmetric model *without* the cubic term, disappears in the presence of the fuzzy sphere in the large-N limit. Coincident fuzzy spheres turn out to be unstable, which implies that the dynamically generated gauge group is U(1).

Keywords: Nonperturbative Effects, Non-Commutative Geometry, Matrix Models.

Contents

1. Introduction

Matrix models are considered as one of the most promising candidates for a nonperturbative formulation of string theories. Indeed some concrete models are proposed as constructive definitions of superstring and M theories [1, 2]. These models are obtained from the dimensional reduction of super Yang-Mills theory in ten dimensions. In a broad sense, such models belong to the class of the so-called dimensionally reduced models [3] (or large-N reduced models), which were studied intensively in the eighties as an equivalent description of large-N gauge theories. Unlike the old models, however, the new models are written in terms of hermitean matrices, and they have manifest supersymmetry, which is expected to have crucial effects on their dynamics.[1]

[1] See ref. [4] for a comprehensive review on these issues.

An important feature of these matrix models is that the space-time is not introduced from the outset, but it emerges dynamically as the eigenvalue distribution of the bosonic matrices. In fact there are certain evidences in the IIB matrix model [2] that *four-dimensional* space-time is generated dynamically [5–8]. In refs. [5–7] the free energy of space-time with various dimensionality has been calculated using the gaussian expansion method, and the free energy turned out to take the minimum value for the four-dimensional space-time. In ref. [8] it was found that the fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ (but not the fuzzy S^2) is a solution to the 2-loop effective action. See also refs. [9–20] for related works on this issue.

By adding a Chern-Simons term to the matrix models, one obtains fuzzy spheres [21] as classical solutions [22–26], and their dynamical properties have been studied in refs. [27–39]. This provides a matrix description of the so-called Myers effect in string theory [40]. The emergence of a fuzzy sphere in matrix models may be regarded as a prototype of the dynamical generation of space-time since it has lower dimensionality than the original dimensionality that the model can actually describe. When k fuzzy spheres coincide, the gauge symmetry enhances from $\mathrm{U}(1)^k$ to $\mathrm{U}(k)$. By expanding the theory around such a solution, one obtains a $\mathrm{U}(k)$ gauge theory on a noncommutative geometry [23]. Therefore the model may also serve as a toy model for the dynamical generation of gauge group, which is expected to occur in the IIB matrix model [41].

In fact one can use the above matrix models to define a regularized field theory on the fuzzy sphere as has been done on a noncommutative torus [42], which enables nonperturbative studies of such theories from first principles [43]. This is motivated from the general expectation that noncommutative geometry provides a crucial link to string theory [44] and quantum gravity [45]. Yet another motivation is to use the fuzzy sphere (or its generalization [46, 47]) as a regularization scheme alternative to the lattice regularization [48]. Unlike the lattice, fuzzy spheres preserve the continuous symmetries of the space-time considered, and hence it is expected to ameliorate the well-known problem concerning chiral symmetry [49–63, 36] and supersymmetry. A challenge in this direction is to remove the effects of noncommutativity of the space-time in the "continuum limit". The fuzzy sphere is also useful in the Coset Space Dimensional Reduction [64, 65], where one can take the compact part of space-time to be a fuzzy coset [66, 67].

Whatever the motivation is, the stability of fuzzy-sphere-like solutions is clearly one of the most important issues. In cases when there are more than one stable solutions, one can identify the true vacuum by comparing the corresponding free energy. This will be important in the dynamical determination of the space-time dimensionality and the gauge group in superstring theory. In the series of papers [68–73], we addressed such issues in various kinds of models using both perturbative calculations and Monte Carlo simulations. In ref. [68] we have studied the dimensionally reduced 3d Yang-Mills model with the Chern-Simons term, which has the fuzzy 2-sphere (S^2) as a classical solution [23]. We have found a first-order phase transition as we vary the coefficient (α) of the Chern-Simons term. For small α the large-N behavior of the model is the same as in the pure Yang-Mills model, whereas for large α a single fuzzy S^2 appears dynamically. In addition we find that the k coincident fuzzy spheres, which are also classical solutions of the same model, cannot be realized as the true vacuum in this model even in the large-N limit.

This implies that the dynamical gauge group is U(1) in this model. In refs. [69, 70, 73] we have extended this work to various matrix models, which incorporate four-dimensional fuzzy manifolds as classical solutions. While the fuzzy S^4 turned out to be unstable [69], we find that the fuzzy CP^2 [70] and the fuzzy $S^2 \times S^2$ [73] are stable at large N although the true vacuum is actually given by the fuzzy S^2. In the latter two cases the gauge group generated dynamically turned out to be U(1) as well. In ref. [72], on the other hand, it has been shown for the first time that gauge groups of higher rank can be realized in the true vacuum by adding a mass term to the 3d Yang-Mills-Chern-Simons model.

The aim of the present paper is to study the impact of supersymmetry on the dynamics of the fuzzy spheres. The simplest 3d supersymmetric model [23] is problematic nonperturbatively since the partition function is divergent [74 – 78]. This leads us to study the 4d supersymmetric model with a cubic term instead. Indeed it turns out that the supersymmetry has striking effects. Unlike the bosonic models, the fuzzy sphere is always stable if the large-N limit is taken in such a way that various correlation functions scale. We also observe an interesting phenomenon that the power-law tail of the eigenvalue distribution, which exists in the supersymmetric models *without* the Chern-Simons term [79, 77], disappears in the presence of the fuzzy sphere in the large-N limit. Coincident fuzzy spheres turn out to be unstable, which implies that the dynamically generated gauge group is U(1) in the present model.

This paper is organized as follows. In section 2 we define the model and discuss its fuzzy sphere solutions. In section 3 we study the phase diagram of the model. In section 4 we study the geometrical structure of the dominant configurations. In section 5 we study coincident fuzzy spheres and discuss the dynamical gauge group. Section 6 is devoted to summary and discussions. In appendix A we explain the algorithm for our Monte Carlo simulations. In appendices B and C we provide the details of perturbative calculations.

2. The model and the fuzzy sphere

The model we study in this paper is defined by the action

$$S = S_{\rm b} + S_{\rm f}\,, \tag{2.1}$$

$$S_{\rm b} = N\ {\rm tr}\left(-\frac{1}{4}\,[\,A_\mu,\,A_\nu\,]^2 + \frac{2}{3}\,i\,\alpha \sum_{i,j,k=1}^{3} \epsilon_{ijk}\,A_i\,A_j\,A_k\right), \tag{2.2}$$

$$S_{\rm f} = -N\ {\rm tr}\left(\,\bar{\psi}_\alpha\,(\Gamma_\mu)_{\alpha\beta}\,[\,A_\mu,\,\psi_\beta\,]\,\right), \tag{2.3}$$

where A_μ ($\mu = 1,2,3,4$) are $N \times N$ traceless hermitean (bosonic) matrices, and $\psi_\alpha, \bar{\psi}_\alpha$ ($\alpha = 1,2$) are $N \times N$ traceless complex (fermionic) matrices. Here and henceforth we assume that repeated Greek indices are summed over all possible integers. The ϵ_{ijk} ($i,j,k = 1,2,3$) is a totally anti-symmetric tensor with $\epsilon_{123} = 1$. The 2×2 matrices Γ_μ are Weyl-projected gamma matrices in four dimensions, and they are given explicitly as

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \tag{2.4}$$

The convergence of the integration over A_μ is a non-trivial issue since the integration region is non-compact. At $\alpha = 0$ the partition function is finite for arbitrary N, as first conjectured by ref. [74] and proved later by ref. [77], and this remains to be the case also for $\alpha \neq 0$ [78]. Moreover, since the fermion determinant of this model is positive semi-definite (See ref. [11] for a proof), the model can be studied by Monte Carlo simulations without confronting the so-called sign problem. The pure super Yang-Mills model ($\alpha = 0$), which may be regarded as the 4d version of the IIB matrix model [2], has been studied intensively [74, 11, 14, 15]. The sign problem does not occur even if one includes the cubic term, which is real.[2]

For $\alpha = 0$ the model has manifest SO(4) symmetry and supersymmetry. The cubic term in (2.2) obviously breaks the SO(4) symmetry down to SO(3). It also breaks supersymmetry, but the effects of breaking is "soft" since the power of A_μ is lower than the quartic term [31]. Therefore one may still anticipate to see peculiar effects of supersymmetry. We repeat that the 3d supersymmetric model, which has been studied perturbatively [23, 31], is actually problematic nonperturbatively since the partition function is divergent [74, 77, 78]. Therefore, the present 4d model is the simplest model that can be studied in order to examine the impact of supersymmetry on the fuzzy sphere dynamics.

Let us then consider the classical solutions of this model. For $\psi = 0$ the equation of motion reads

$$\begin{aligned} &[A_\nu, [A_\nu, A_i]] + i\,\alpha \sum_{j,k=1}^{3} \epsilon_{ijk}\,[A_j, A_k] = 0 \quad \text{for } i = 1,2,3\,, \\ &[A_\nu, [A_\nu, A_4]] = 0\,. \end{aligned} \tag{2.5}$$

Apart from the solution given by commuting matrices, which exists also for $\alpha = 0$, we have the fuzzy S^2 solution given by

$$\begin{cases} A_i^{(\mathrm{S}^2)} = \alpha\, L_i^{(N)} \quad \text{for } i = 1,2,3\,, \\ A_4^{(\mathrm{S}^2)} = 0\,, \end{cases} \tag{2.6}$$

where $L_i^{(r)}$ $(i = 1,2,3)$ represents the r-dimensional *irreducible* representation of the SU(2) Lie algebra

$$\left[L_i^{(r)}, L_j^{(r)}\right] = i\,\alpha\,\epsilon_{ijk}\,L_k^{(r)}\,. \tag{2.7}$$

The solution $A_\mu^{(\mathrm{S}^2)}$ satisfies

$$\sum_{i=1}^{3} \left(A_i^{(\mathrm{S}^2)}\right)^2 = \frac{1}{4}\,(N^2 - 1)\,\alpha^2\,\mathbf{1}_N\,, \tag{2.8}$$

which implies that the "radius" of the fuzzy sphere is given by

$$\rho = \frac{1}{2}\,\alpha\,\sqrt{N^2 - 1}\,. \tag{2.9}$$

We consider more general solutions in section 5.

[2]This is in contrast to the Chern-Simon term in ordinary gauge theories in euclidean space-time, which is purely imaginary. Note that the coefficient α in (2.2) should be chosen to be real in order for fuzzy sphere solutions to exist.

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3. Phase diagram

3.1 Monte Carlo results

In this section we calculate various quantities by Monte Carlo simulation and study the phase diagram of the model (2.1). We show results obtained by using the fuzzy sphere $A_\mu^{(\mathrm{S}^2)}$ as the initial configuration, but we have checked that the result is the same for other initial configurations such as $A_\mu = 0$ or some randomly generated configurations. For brevity we introduce the notation

$$F_{\mu\nu} = i\,[A_\mu, A_\nu]\,, \tag{3.1}$$

$$M = \frac{2}{3\,N}\, i \sum_{i,j,k=1}^{3} \epsilon_{ijk}\ \mathrm{tr}\,(\,A_i\,A_j\,A_k\,)\,. \tag{3.2}$$

We note that there is an exact result

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(F_{\mu\nu})^2 \right\rangle + 3\,\alpha\,\langle M \rangle = 6\left(1 - \frac{1}{N^2}\right), \tag{3.3}$$

which can be derived as in the bosonic case [68]. This result has been used to check our code for the simulation.

By performing one-loop calculation around the fuzzy sphere $A_\mu^{(\mathrm{S}^2)}$, we obtain the leading large-N behaviors as (See appendix C for the details)

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(F_{\mu\nu})^2 \right\rangle \simeq \frac{1}{2}\,\tilde{\alpha}^4 + 6\,, \tag{3.4}$$

$$\frac{1}{\sqrt{N}}\,\langle M \rangle \simeq -\frac{1}{6}\,\tilde{\alpha}^3 + 0\,, \tag{3.5}$$

$$\frac{1}{N}\left\langle \frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2 \right\rangle \simeq \frac{1}{4}\,\tilde{\alpha}^2 + 0\,, \tag{3.6}$$

where we have introduced the rescaled parameter

$$\tilde{\alpha} = \alpha\,\sqrt{N}\,. \tag{3.7}$$

In the r.h.s. of eqs. (3.4) $\sim$ (3.6) the first term represents the classical result, and the second term represents the one-loop correction.

In figure 1 we plot the results for $\langle \frac{1}{N}\ \mathrm{tr}\,(F_{\mu\nu})^2 \rangle$ and $\frac{1}{\sqrt{N}}\langle M \rangle$ obtained by Monte Carlo simulations. We find that Monte Carlo data agree with the one-loop results even at $\tilde{\alpha} = 0$. This is rather surprising since the expansion parameter in the perturbative calculation (at finite N) is $\frac{1}{\alpha^4}$. As we will see shortly, however, the system actually changes its behavior at $\tilde{\alpha} \propto \frac{1}{\sqrt{N}}$, and the agreement in figure 1 below that point should rather be considered as accidental.

Let us then consider the quantity $\langle \frac{1}{N}\ \mathrm{tr}\,(A_\mu)^2 \rangle$, which we have postponed since it involves a subtle issue. At $\alpha = 0$ this quantity is actually divergent even for finite N, as first observed in numerical studies [79] and further confirmed by ref. [14]. (Ref. [77] provides some analytical explanation.) On the other hand, from perturbative calculations

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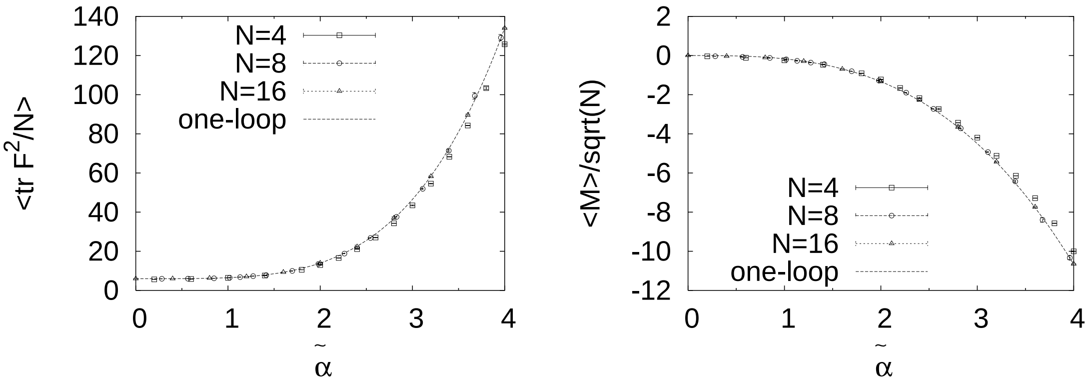

Figure 1: Various observables are plotted against $\tilde{\alpha} = \alpha\sqrt{N}$ for $N = 4, 8, 16$. The dashed lines represent the one-loop results at large N.

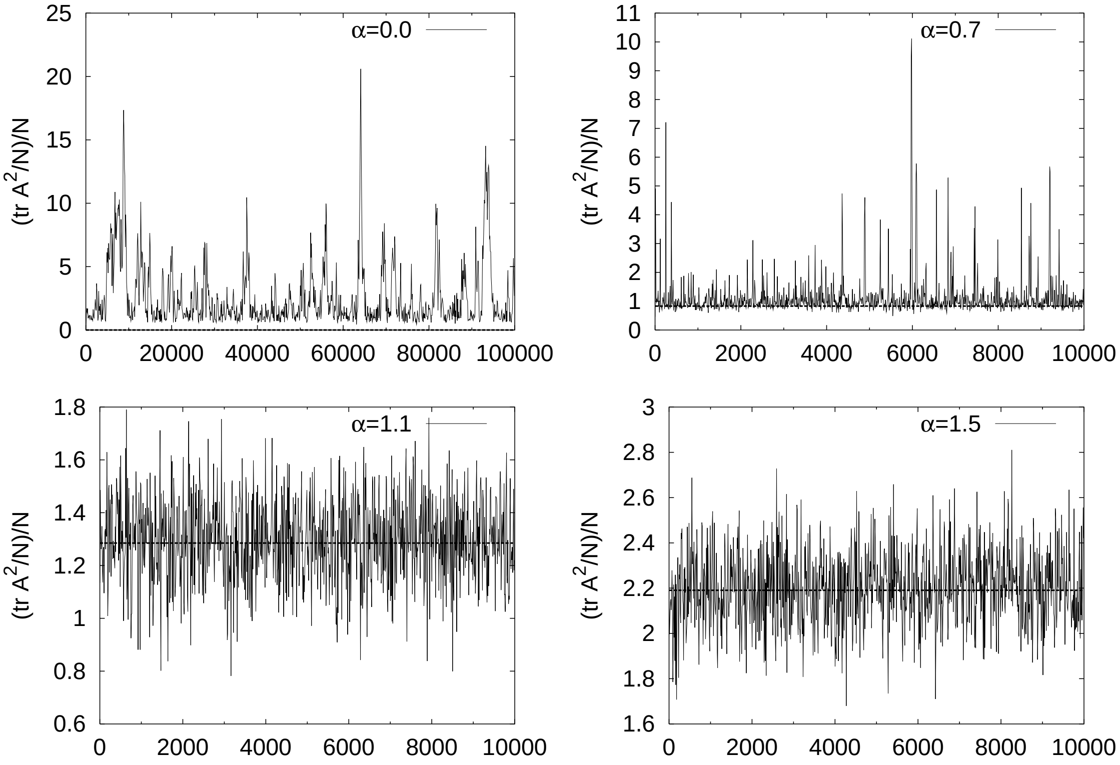

Figure 2: The history of $(\frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2)/N$ is shown for various α at $N = 4$. The horizontal lines represent the one-loop results for $\frac{1}{N}\langle\frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2\rangle$.

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around the fuzzy sphere, we obtain a finite result for finite N (See eqs. (C.6) and (C.7)). In order to clarify the situation, let us first look at the history of $\frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2$, which is plotted in figure 2 for various α at $N = 4$. The horizontal axis represents the number of "trajectories" in the hybrid Monte Carlo algorithm (See appendix A.) with the parameters $\nu = 100$ and $\Delta\tau = 0.01$. For $\alpha = 0$ the history has a lot of spikes, and these spikes are responsible for the divergence of $\left\langle\frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2\right\rangle$. As we increase α the spikes become less

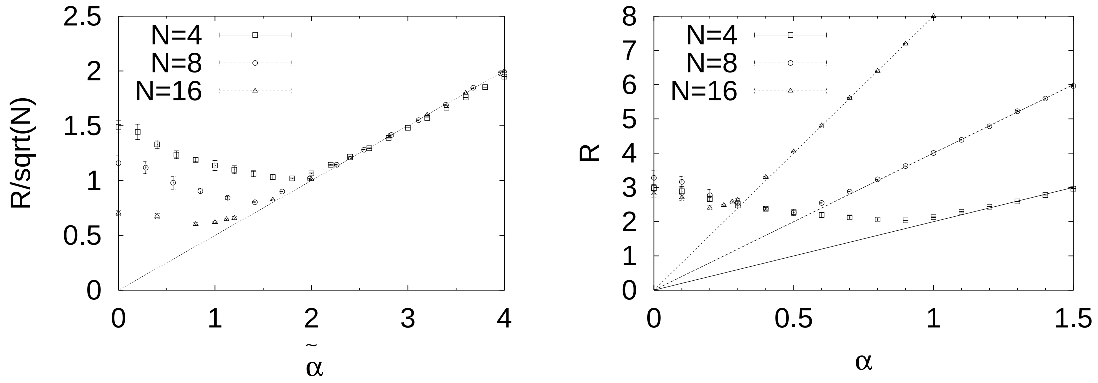

Figure 3: On the left, Monte Carlo results for $\frac{R}{\sqrt{N}}$ are plotted against $\tilde{\alpha}$ for $N = 4, 8, 16$. On the right the same data are plotted in a different scale. The straight lines represent the classical result $R = \frac{1}{2}\,\tilde{\alpha}\,\sqrt{N}$ for the fuzzy sphere solution.

frequent, and their height gets lowered. At $\alpha \gtrsim 1.1$ the history looks quite regular. We also looked at the history at larger N, and find that it becomes regular (no spikes) for $\alpha \gtrsim 0.5$ at $N = 8$ and for $\alpha \gtrsim 0.3$ at $N = 16$. The transition point $\alpha_{\rm tr}$ is roughly consistent with $\alpha_{\rm tr} \propto \frac{1}{N}$ (i.e., $\tilde{\alpha}_{\rm tr} \propto \frac{1}{\sqrt{N}}$).

According to ref. [78], switching on α does not change the convergence properties of the matrix integrals. Therefore, unless there is some special mechanism for canceling the leading divergence, we get $\left\langle \frac{1}{N}\,{\rm tr}\,(A_\mu)^2 \right\rangle = \infty$ even for $\alpha \neq 0$. The reason why we get finite results by perturbative calculations should then be that such a calculation only include the region of the configuration space near the fuzzy sphere solution so that it does not take into account the configurations that have large $\frac{1}{N}\,{\rm tr}\,(A_\mu)^2$, which may actually contribute crucially to the vacuum expectation value of that quantity.

One can, however, define a finite quantity [14]

$$R = \left\langle \sqrt{\frac{1}{N}\,{\rm tr}\,(A_\mu)^2} \right\rangle , \tag{3.8}$$

which is finite at $\alpha = 0$ and behaves as O(1) at large N [11] [3] in the present parametrization of the action. Due to the argument of ref. [78], this quantity is finite also for $\alpha \neq 0$. We plot the Monte Carlo results for $R/\sqrt{N}$ in figure 3 on the left as a function of $\tilde{\alpha}$ for $N = 4$, 8 and 16. At large $\tilde{\alpha}$ the data agree very well with the classical result $R = \frac{1}{2}\,\tilde{\alpha}\,\sqrt{N}$ for the fuzzy sphere solution.

In figure 3 on the right we plot R against α. We expect that R approaches a finite value in the large-N limit for each α in the small-α regime, and our data are roughly consistent with this picture. If we assume the transition to take place at the point where the fuzzy sphere result $R \simeq \frac{1}{2}\,\tilde{\alpha}\,\sqrt{N}$ becomes comparable to the pure super Yang-Mills behavior $R \simeq {\rm O}(1)$, the transition point should be $\tilde{\alpha} \propto \frac{1}{\sqrt{N}}$, which roughly agrees with the point $\tilde{\alpha}_{\rm tr}$, where the spikes in figure 2 get suppressed.

[3] Strictly speaking, ref. [11] studies a slightly different observable, but the large-N behavior is expected to be qualitatively the same.

Thus we conclude that the system actually undergoes some transition at $\tilde{\alpha} \propto \frac{1}{\sqrt{N}}$, below which the system behaves similarly to the pure super Yang-Mills model ($\alpha = 0$). The quantities in figure 1 are insensitive to the transition since their behavior in the pure super Yang-Mills phase just happens to be the same as in the fuzzy sphere phase. Note that $\langle M \rangle_{\alpha=0} = 0$ due to parity symmetry $A_i \mapsto -A_i$, whereas from the perturbative expansion around the fuzzy sphere, one finds that the one-loop contribution to $\langle M \rangle$ is absent due to supersymmetry,[4] and as a consequence the result (C.9) has a smooth extrapolation to $\alpha = 0$, which agrees with $\langle M \rangle_{\alpha=0} = 0$. Since $\langle M \rangle$ and $\langle \frac{1}{N} \operatorname{tr} F^2 \rangle$ are related to each other through the exact result (3.3), the agreement of the former propagates to that of the latter. Thus the success of one-loop results for these quantities in the small-$\tilde{\alpha}$ regime does not mean that we are still in the fuzzy sphere phase, but it simply means that these quantities are insensitive to the transition from the fuzzy sphere phase to the pure super Yang-Mills phase.

The above results are in striking contrast to those obtained in the bosonic model [68], where we observed that the fuzzy sphere becomes unstable at some finite $\tilde{\alpha}$, and various quantities show a hysteresis behavior, implying a first order phase transition.

3.2 Theoretical understanding based on the effective action

In the previous section we observed that the fuzzy sphere is stable in the large-N limit at any finite $\tilde{\alpha}$ unlike the bosonic model. Here we would like to provide some theoretical understanding of the striking difference between the bosonic and supersymmetric cases based on the one-loop effective action. For that purpose let us consider a one-parameter family of configurations given by

$$\begin{cases} A_i = \beta \, L_i^{(N)} \qquad \text{for } i = 1, 2, 3 \, , \\ A_4 = 0 \, , \end{cases} \tag{3.9}$$

where the fuzzy sphere solution (2.6) corresponds to $\beta = \alpha$. The one-loop effective action around (3.9) can be calculated along the line described in appendix B, and we get the result at large N as

$$\frac{1}{N^2} W_{1-\text{loop}}^{(\beta)} = \left(\frac{1}{8} \tilde{\beta}^4 - \frac{1}{6} \tilde{\alpha} \tilde{\beta}^3 \right) - \log N \, , \tag{3.10}$$

where $\tilde{\beta} = \beta \sqrt{N}$. The one-loop effective action has a minimum at $\tilde{\beta} = \tilde{\alpha}$ for arbitrary $\tilde{\alpha}$.

In analogous calculations in the bosonic models [68, 70, 73], the one-loop contribution gives rise to a term proportional to $\log \tilde{\beta}$. Due to this term the (local) minimum disappears below some critical $\tilde{\alpha}$, which indeed agrees well with the Monte Carlo results. In the present supersymmetric case, the $\tilde{\beta}$-dependent one-loop term is absent due to supersymmetry. Thus we can understand the qualitative difference between the bosonic case and the supersymmetric case observed in Monte Carlo simulations.

[4]Note that the cubic term, which breaks supersymmetry softly, does not appear in the relevant one-loop calculation. This is not the case, however, at higher loop calculations.

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4. Geometrical structure

In this section we study the geometrical structure of the dominant configurations in the supersymmetric model. For that purpose we consider the "Casimir operator"

$$Q = (A_\mu)^2, \tag{4.1}$$

and define its eigenvalue distribution $f(x)$ as

$$f(x) = \frac{1}{N} \sum_{j=1}^{N} \Big\langle \delta(x - \lambda_j) \Big\rangle , \tag{4.2}$$

where λ_j $(j = 1, 2, \ldots, N)$ represent the eigenvalues of Q. Let us note that $\langle \frac{1}{N} \operatorname{tr} (A_\mu)^2 \rangle$ discussed in the previous section is related to $f(x)$ as

$$\Big\langle \frac{1}{N} \operatorname{tr} (A_\mu)^2 \Big\rangle = \Big\langle \frac{1}{N} \operatorname{tr} Q \Big\rangle = \Big\langle \frac{1}{N} \sum_{j=1}^{N} \lambda_j \Big\rangle = \int_0^\infty x\, f(x)\, dx\,. \tag{4.3}$$

In figure 4 we plot the eigenvalue distribution for the same set of α and N as in figure 2.

4.1 Power-law tail

The results for $\alpha = 0$ reproduce the power-law behavior

$$f(x) \propto x^{-2}\,, \tag{4.4}$$

at large x, which has been first discovered in ref. [79] and studied also in ref. [80]. This is related to the divergence of $\langle \frac{1}{N} \operatorname{tr} (A_\mu)^2 \rangle$ discussed in section 3.1. In fact this quantity is known to diverge *logarithmically* [79], which explains the power in (4.4) due to (4.3).

At $\alpha = 0.7$ the magnitude of the power-law tail becomes much weaker, but our data are consistent with the existence of the power-law tail. At $\alpha \gtrsim 1.1$ the power-law tail becomes hardly visible, which corresponds to the disappearance of the spikes in the history of $\frac{1}{N} \operatorname{tr} (A_\mu)^2$ seen in figure 2. If we assume that $\langle \frac{1}{N} \operatorname{tr} (A_\mu)^2 \rangle = \infty$ for $\alpha \neq 0$, as we argued in the previous section, the power-law tail should be still there, but simply hidden by the main contribution coming from the fuzzy-sphere-like configurations.[5] Considering that those configurations are enhanced by the Boltzmann weight $e^{\text{const.}\tilde{\alpha}^4 N^2}$ at large $\tilde{\alpha}$ compared with the configurations that give the power-law tail, we expect that the magnitude of the power-law tail decreases as $e^{-\text{const.}\tilde{\alpha}^4 N^2}$. Therefore the power-law tail is expected to disappear completely if we take the large-N limit at fixed $\tilde{\alpha}$.

[5] Within perturbative expansion around the fuzzy sphere configuration, we expect to obtain a distribution which decays faster since the n-th moment $\int dx\, x^n f(x) = \frac{1}{N} \langle \operatorname{tr} \{(A_\mu)^2\}^n \rangle$ can be calculated as a finite quantity to all orders.

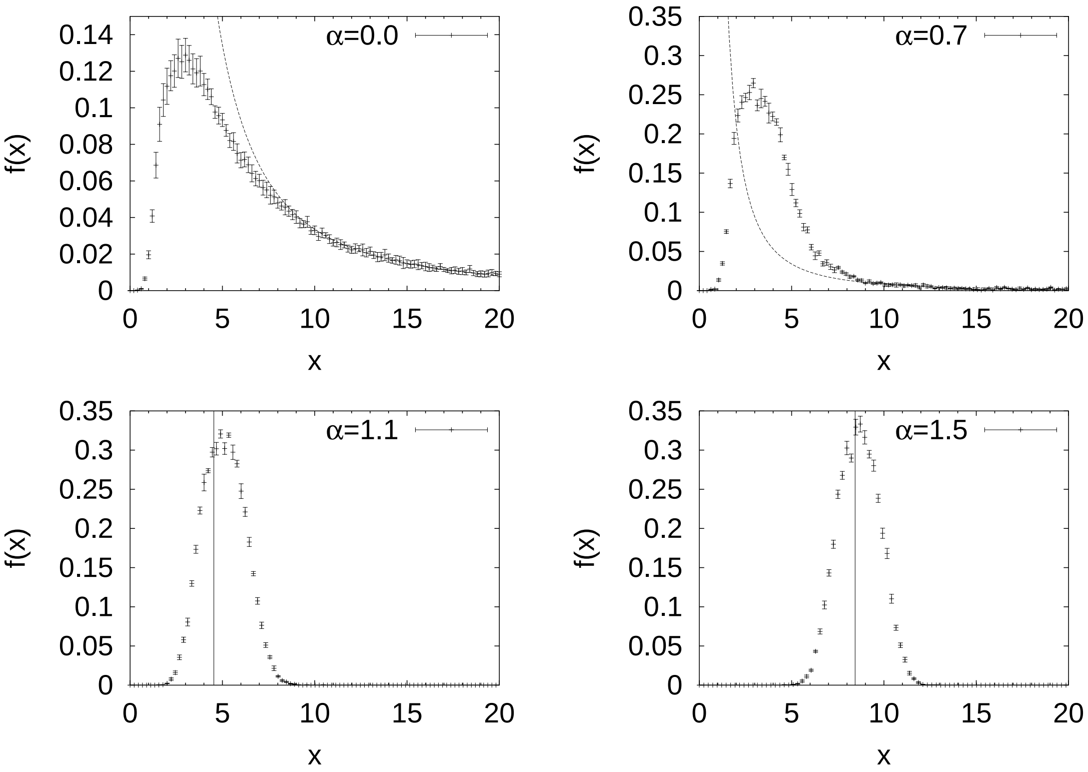

Figure 4: The plot of the eigenvalue distribution $f(x)$ of the Casimir operator Q for various α at $N = 4$. The dashed lines in the two upper figures represent a fit to the power-law behavior (4.4), and the vertical lines in the two lower figures represent the classical results (δ-function) for the single fuzzy sphere solution.

4.2 Spherical geometry

In order to clarify the geometrical structure of the fuzzy-sphere-like configurations, we decompose the Casimir operator Q as $Q = Q^{(123)} + Q^{(4)}$, where

$$Q^{(123)} = \sum_{i=1}^{3} (A_i)^2 \,, \tag{4.5}$$

$$Q^{(4)} = (A_4)^2 \,, \tag{4.6}$$

and calculate the eigenvalue distribution for $Q^{(123)}$ and $Q^{(4)}$, which we denote as $f^{(123)}(x)$ and $f^{(4)}(x)$, respectively. Figure 5 shows the result for $\alpha = 1.5$ at $N = 4$.

The figure on the right shows that the eigenvalues of $Q^{(4)}$ is quite small, which can be understood from the one-loop result (C.7) for $\left\langle \frac{1}{N} \operatorname{tr} (A_4)^2 \right\rangle_{1-\text{loop}}$, which vanishes as $\mathrm{O}(\frac{1}{N^2} \log N)$ in the large-N limit with fixed $\tilde{\alpha}$. As a consequence the distribution $f^{(123)}(x)$ shown on the left of the figure 5 is almost identical to $f(x)$ shown on the bottom right of figure 4. We also find that $f^{(123)}(x)$ is peaked around the classical result. Thus we confirm that the dominant configurations indeed have the geometry of a 2-sphere.

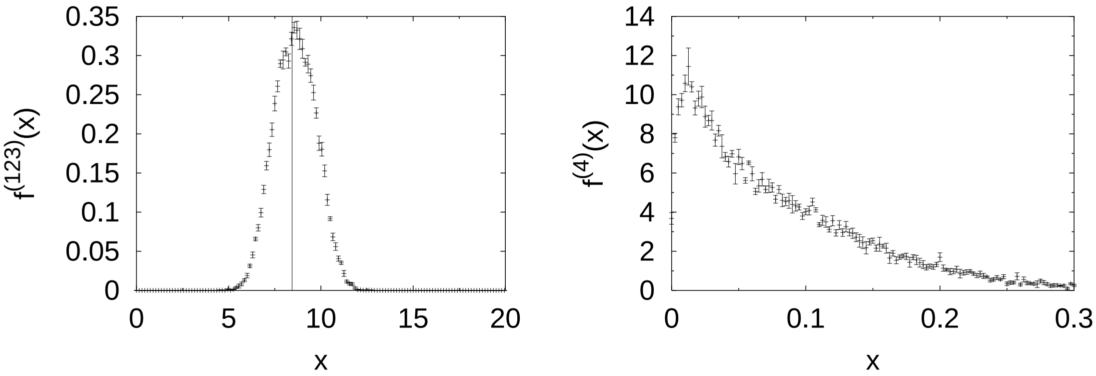

Figure 5: The functions $f^{(123)}(x)$ and $f^{(4)}(x)$, which are the eigenvalue distributions of the operator $Q^{(123)}$ and $Q^{(4)}$, respectively, are plotted for $\alpha = 1.5$ at $N = 4$. The classical result for the single fuzzy sphere solution is represented by the vertical line (δ-function) in the left figure.

At $\alpha = 0$ the distribution $f(x)$ has an empty region around $x = 0$. Similar behavior has been observed also in the bosonic model [68], and it can be understood by the uncertainty principle. Therefore the geometrical structure of the dominant configurations at $\alpha = 0$ should rather be considered as that of a solid ball.

5. Dynamical gauge group

In fact the equation of motion (2.5) has a class of solutions of the form

$$\begin{cases} A_i^{(k\mathrm{S}^2)} = \alpha \left(L_i^{(n)} \otimes \mathbf{1}_k \right) \qquad \text{for } i = 1, 2, 3\,, \\ A_4^{(k\mathrm{S}^2)} = 0\,, \end{cases} \tag{5.1}$$

where $N = n\,k$. These solutions represent the k coincident fuzzy S^2, and the Casimir operator Q takes the value

$$Q = \frac{1}{4}\,(n^2 - 1)\,\alpha^2\,\mathbf{1}_N\,, \tag{5.2}$$

meaning that the "radius" of the fuzzy spheres is given by $\rho = \frac{1}{2}\,\alpha\,\sqrt{n^2 - 1} \simeq \frac{1}{2k}\,\alpha\,N$, which becomes smaller as k increases. By expanding the theory around such a configuration, one obtains noncommutative Yang-Mills theory with the $\mathrm{U}(k)$ gauge group [23].

In order to study the stability of such a configuration, we perform Monte Carlo simulation for $N = 8$ and $\alpha = 1.0$ using the $k = 2$ coincident fuzzy spheres as the initial configuration. In figure 6 we plot the history of the eigenvalues of Q. The horizontal axis represents the number of "trajectories" in the hybrid Monte Carlo algorithm (See appendix A.) with the parameters $\nu = 100$ and $\Delta\tau = 0.0001$. Thus the $k = 2$ multi fuzzy sphere configuration is unstable and decays into the single fuzzy sphere.

This phenomenon can be understood by considering the free energy, which is calculated in appendix B up to one-loop. At large N the result reads

$$\frac{1}{N^2}\,W^{(k)}_{1-\text{loop}} = -\frac{1}{24\,k^2}\,\tilde{\alpha}^4 - \log N\,. \tag{5.3}$$

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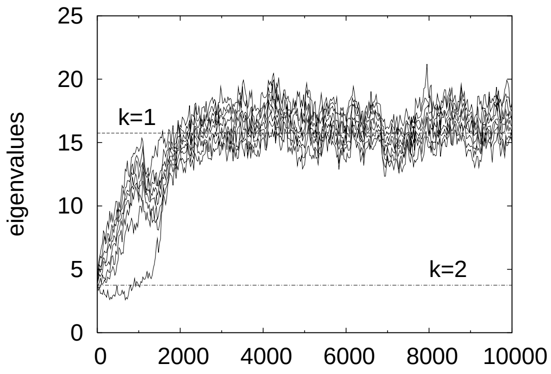

Figure 6: The history of the eigenvalues of the Casimir operator Q obtained for $N = 8$, $\alpha = 1.0$ using the $k = 2$ solution as the initial configuration. The horizontal lines represent classical results for the single fuzzy sphere ($k = 1$) and the two coincident fuzzy-spheres ($k = 2$), respectively.

Since the one-loop term represented by the second term is independent of k, we find that the free energy takes the smallest value for $k = 1$.

Although the conclusion concerning the dynamical gauge group is the same as in the bosonic models [68, 70, 73], we note that the reasoning is different. In the bosonic models the one-loop term in the free energy has the $-\log k^2$ term, which actually favors large k. However, if one decreases $\tilde{\alpha}$ so that the one-loop term becomes more important, the fuzzy sphere solutions disappear before the free energy starts to favor $k > 1$. In the present supersymmetric case, the fuzzy sphere solutions remain to be there, but due to the absence of the $-\log k^2$ term, the $k = 1$ solution is always favored.

6. Conclusion

In this paper we have studied the dimensionally reduced 4d super Yang-Mills model with an extra Chern-Simons term, which incorporates fuzzy spheres as classical solutions. We have found that supersymmetry indeed has substantial effects on the dynamics of fuzzy spheres.

While the observables that appear in the action change continuously as we vary α, the model actually possesses two distinctive phases, which is demonstrated by a well-defined observable $\left\langle \sqrt{\frac{1}{N}\,\mathrm{tr}\,(A_\mu)^2} \right\rangle$. The tail of the eigenvalue distribution changes drastically as one moves from one phase to the other. In the pure super Yang-Mills phase we observe the same power-law tail as the one known for $\alpha = 0$, but it disappears in the fuzzy sphere phase in the large-N limit. This allows us to identify the critical point quite accurately. From our Monte Carlo data up to $N = 16$ and some theoretical considerations, we speculate that the transition point $\tilde{\alpha}_{\mathrm{tr}}$ goes to zero as $\tilde{\alpha}_{\mathrm{tr}} = \mathrm{O}\left(\frac{1}{\sqrt{N}}\right)$.

Our results are in sharp contrast to the results obtained for analogous bosonic models [68, 70, 73], where the fuzzy sphere becomes unstable at some finite critical $\tilde{\alpha}$. A strong first-order phase transition has been observed in the bosonic models, and the (lower) critical point obtained by Monte Carlo simulation can be reproduced very well from the one-loop effective action. In the present paper we have shown that the one-loop term in the effective action vanishes up to an irrelevant constant due to supersymmetry. This explains our observation that the fuzzy sphere is stable down to vanishingly small $\tilde{\alpha}$ at large N.

We have also studied the dynamical generation of the gauge group. In the bosonic case the quantum instability of the fuzzy spheres was an obstacle in obtaining a non-trivial gauge group in the true vacuum. In the present supersymmetric case, this instability is gone, but we also lost the one-loop term in the free energy for the coincident fuzzy spheres, which favors higher multiplicity. As a result, we obtain the U(1) gauge group again. We note, however, that this conclusion applies only to the fuzzy sphere phase, and whether we can obtain a nontrivial gauge group in supersymmetric models *without a Chern-Simons term* such as the IIB matrix model [2] is still an interesting open question.

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A. Details of the Monte Carlo simulation

In this section we explain the algorithm used for our simulation. Our algorithm is similar to the one adopted in ref. [11], but the crucial difference is that we make the Metropolis reject/accept procedure at the end of each trajectory. In the previous algorithm there was a systematic error due to discretization required for solving Hamilton's equation, and the step size $\Delta\tau$ for the discretization had to be sent to zero. In the present algorithm we do not need such an extrapolation. Another difference is that we do not use the noisy estimator for estimating the r.h.s. of Hamilton's equation since it causes some systematic error. Instead we invert the Dirac operator directly using the LU decomposition. Each of the two modifications increases the computational effort for making one trajectory (for fixed parameters in the algorithm) from $\mathrm{O}(N^5)$ to $\mathrm{O}(N^6)$, which is the price we have to pay to make the algorithm "exact".

The "exact" algorithm is essentially the hybrid Monte Carlo (HMC) algorithm, which is used in studying the large-N behavior of the phase quenched version of the IIB matrix model [12]. In that case the one-loop approximation has been used to decrease the computational effort from $\mathrm{O}(N^6)$ to $\mathrm{O}(N^3)$. The hybrid algorithms are standard in full QCD simulations, and it is useful also in simulating matrix models as demonstrated in refs. [11, 12]. If we had used the Metropolis algorithm [14] in the present model, for instance, the computational effort would have been $\mathrm{O}(N^8)$. Note, however, that simulating matrix models is generally harder than simulating field theories due to the non-local nature of the interaction. Even in the bosonic case, the computational effort is at least $\mathrm{O}(N^3)$, which grows faster than the number of d.o.f., which is $\mathrm{O}(N^2)$.

Let us first recall an explicit form of the fermion determinant derived in ref. [11]. We define a complete basis for the general complex $N \times N$ matrices as

$$(t^a)_{ij} = \delta_{i\,i_a}\delta_{j\,j_a} \qquad (a = 1, 2, \ldots, N^2)\,, \tag{A.1}$$

where i_a, j_a are integers within the range $1 \le i_a, j_a \le N$ satisfying

$$a = N\,(i_a - 1) + j_a\,. \tag{A.2}$$

By taking into account that the fermionic matrices ψ_α and $\bar{\psi}_\alpha$ are traceless, integration over the fermionic matrices yields the fermion determinant $\det \mathcal{M}$, where the $2(N^2-1) \times 2(N^2-1)$ matrix $\mathcal{M}$ is given by

$$\mathcal{M}_{a\alpha,b\beta} = \mathcal{M}'_{a\alpha,b\beta} - \mathcal{M}'_{N^2\alpha,b\beta}\,\delta_{i_a j_a} - \mathcal{M}'_{a\alpha,N^2\beta}\,\delta_{i_b j_b}\,. \tag{A.3}$$

Here the $2N^2 \times 2N^2$ matrix $\mathcal{M}'$ is defined as

$$\mathcal{M}'_{a\alpha,b\beta} = (\Gamma_\mu)_{\alpha\beta}\ \mathrm{tr}\,(t^a[A_\mu, t^b])\,. \tag{A.4}$$

The effective action for A_μ can be written as

$$S_{\mathrm{eff}}[A] = S_{\mathrm{b}}[A] - \log\,\mathrm{Det}\,\mathcal{M}[A]\,. \tag{A.5}$$

Following the idea of the hybrid Monte Carlo algorithm, we introduce auxiliary bosonic hermitean matrices P_μ and consider the action

$$S_{\mathrm{HMC}}[P, A] = \frac{1}{2}\ \mathrm{tr}\,(P_\mu)^2 + S_{\mathrm{eff}}[A]\,. \tag{A.6}$$

Since P_μ does not couple to A_μ, we retrieve the original model trivially by integrating out P_μ. We regard the action $S_{\mathrm{HMC}}[P, A]$ as the hamiltonian of a classical system described by $A_\mu(\tau)$ and its conjugate momentum $P_\mu(\tau)$, where τ denotes the fictitious time of the classical system. Then as an update procedure, we may take the old configuration (P, A) as the initial configuration $(P(0), A(0))$ and solve Hamilton's equation

$$\frac{d(A_\mu)_{ij}}{d\tau} = \frac{\partial S_{\mathrm{HMC}}}{\partial (P_\mu)_{ij}} = (P_\mu)_{ji}\,, \tag{A.7}$$

$$\begin{aligned}\frac{d(P_\mu)_{ij}}{d\tau} &= -\frac{\partial S_{\mathrm{HMC}}}{\partial (A_\mu)_{ij}} = -\frac{\partial S_{\mathrm{eff}}}{\partial (A_\mu)_{ij}} \\ &= N\left(-[A_\nu, [A_\mu, A_\nu]] + 2\,i\,\alpha\,\epsilon_{\mu\nu\rho} A_\nu A_\rho\right)_{ji} - \mathrm{Tr}\left(\mathcal{M}^{-1}\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ij}}\right)\end{aligned} \tag{A.8}$$

for a finite fictitious time T (this defines "one trajectory") to obtain $(P(T), A(T))$. The symbol Tr in (A.8) denotes a trace over the $2\,(N^2-1)$-dimensional index, and the derivative $\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ij}}$ is given explicitly by

$$\frac{\partial \mathcal{M}_{a\alpha,b\beta}}{\partial (A_\mu)_{ij}} = -(\Gamma_\mu)_{\alpha\beta}\left([t^b, t^a]\right)_{ji}. \tag{A.9}$$

Since the trace of A_μ is not conserved during the evolution, we subtract the trace part $A'_\mu = A_\mu(T) - \left\{\frac{1}{N}\,\mathrm{tr}\,A_\mu(T)\right\}\mathbf{1}$. Thus we obtain the updated configuration (P', A'), where $P'_\mu = P_\mu(T)$. Due to the hamiltonian conservation, this update procedure preserves the action $S_{\mathrm{HMC}}[P, A]$. Using also the fact that the transition between (P, A) and $(-P', A')$ is reversible, one can readily verify the detailed balance. After each trajectory, we update the momentum P_μ fixing A_μ, which can be done by simply generating gaussian variables since the P_μ-dependent part of the action (A.6) is gaussian. This procedure is necessary to avoid the ergodicity problem.

In actual calculations we have to discretize Hamilton's equation (A.8). The reversibility of the time evolution can be preserved by using the so-called leap-frog discretization, but the hamiltonian conservation is inevitably violated. However, we may accept the configuration (P', A') as the updated configuration with the probability $\max(1, e^{-\Delta S_{\mathrm{HMC}}})$, where $\Delta S_{\mathrm{HMC}} = S_{\mathrm{HMC}}[P', A'] - S_{\mathrm{HMC}}[P, A]$, and duplicate the old configuration when rejected. By adding such a Metropolis accept/reject procedure, we can preserve the detailed balance. The step size $\Delta\tau$ for the time evolution should be small enough to keep the acceptance rate reasonably high. Discretized Hamilton's equation is given by

$$\begin{aligned}
(P_\mu^{(1/2)})_{ij} &= (P_\mu^{(0)})_{ij} - \frac{\Delta\tau}{2}\,\frac{dS_{\mathrm{eff}}}{d(A_\mu)_{ij}}(A_\mu^{(0)})\,, \\
(A_\mu^{(1)})_{ij} &= (A_\mu^{(0)})_{ij} + \Delta\tau\,(P_\mu^{(1/2)})_{ji}\,, && \text{(A.10)} \\
(P_\mu^{(n+1/2)})_{ij} &= (P_\mu^{(n-1/2)})_{ij} - \Delta\tau\,\frac{dS_{\mathrm{eff}}}{d(A_\mu)_{ij}}(A_\mu^{(n)})\,, \\
(A_\mu^{(n+1)})_{ij} &= (A_\mu^{(n)})_{ij} + \Delta\tau\,(P_\mu^{(n+1/2)})_{ji}\,, && \text{(A.11)} \\
(P_\mu^{(\nu)})_{ij} &= (P_\mu^{(\nu-1/2)})_{ij} - \frac{\Delta\tau}{2}\,\frac{dS_{\mathrm{eff}}}{d(A_\mu)_{ij}}(A_\mu^{(\nu)})\,, && \text{(A.12)}
\end{aligned}$$

where $n = 1, 2, \cdots, \nu - 1$ and $T = \nu\,\Delta\tau$, and we have introduced the short-hand notation $P_\mu^{(r)} = P_\mu(r\,\Delta\tau)$ and $A_\mu^{(r)} = A_\mu(r\,\Delta\tau)$. At each step of the "Molecular Dynamics", we have to calculate the inverse $\mathcal{M}^{-1}$, and at the end of each trajectory, we have to calculate $\det\mathcal{M}$. These are the dominant part of the numerical calculation, and it requires a CPU time of the order of $\mathrm{O}(N^6)$.

The hybrid Monte Carlo algorithm involves two parameters T and $\Delta\tau$, which can be optimized in such a way that the computational effort for obtaining one statistically independent configuration is minimized. The optimization can be done in a standard way [12]. First we fix T and optimize $\Delta\tau$ so that the effective speed of motion in the configuration space, which is given by the acceptance rate times $\Delta\tau$, is maximized. Using the $\Delta\tau$ optimized for each T, we minimize the autocorrelation time (in units of "Molecular Dynamics step") with respect to T. For instance, at $N = 16$ and $\alpha = 0.0$ we obtain the optimal values $\Delta\tau \sim 0.006$ and $T \sim 1.0$.

B. One-loop free energy

In this section we formulate the perturbation theory around fuzzy sphere solutions, and derive the one-loop free energy. We decompose A_μ, ψ and $\bar\psi$ into the classical background and the fluctuation as

$$A_\mu = X_\mu + \tilde{A}_\mu \,, \tag{B.1}$$

$$\psi = \chi + \tilde{\psi} \,, \qquad \bar{\psi} = \bar{\chi} + \tilde{\bar{\psi}} \,, \tag{B.2}$$

and obtain the free energy around the classical solutions by integrating over $\tilde{A}_\mu$, $\tilde{\psi}$ and $\tilde{\bar{\psi}}$ perturbatively. Here we take the classical solution to be the k coincident fuzzy spheres $X_\mu = A_\mu^{(k\mathrm{S}^2)}$, $\chi = \bar{\chi} = 0$, which includes the single fuzzy sphere as a special case $k = 1$.

In order to remove the zero modes associated with the SU(N) invariance, we introduce the gauge fixing term and the corresponding ghost term

$$S_{\text{g.f.}} = -\frac{N}{2} \, \mathrm{tr} \, [\, X_\mu, \, A_\mu \,]^2 \,, \tag{B.3}$$

$$S_{\text{ghost}} = -N \ \mathrm{tr} \ (\, [\, X_\mu, \bar{c}\,] \, [\, A_\mu, c\,] \,) = N \ \mathrm{tr} \ (\, \bar{c} \, [\, X_\mu, \, [\, A_\mu, \, c\,]\,]\,) \ , \tag{B.4}$$

where c and $\bar{c}$ are the ghost and anti-ghost fields respectively.

The total action $S_{\text{total}} = S + S_{\text{g.f.}} + S_{\text{ghost}}$ can be written as

$$S_{\text{total}} = S_{\text{cl}} + S_{\text{kin}} + S_{\text{int}} \ , \tag{B.5}$$

$$S_{\text{cl}} = N \ \mathrm{tr} \left(-\frac{1}{4} \, [X_\mu, X_\nu]^2 + \frac{2}{3} \, i \, \alpha \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, X_i \, X_j \ X_k \right) \ , \tag{B.6}$$

$$\begin{aligned} S_{\text{kin}} = N \ \mathrm{tr} \Bigg(& -[\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, X_\nu] + i \, \alpha \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, [\, \tilde{A}_i, \, \tilde{A}_j \,] \, X_k \, - \\ & - \frac{1}{2} \, [X_\mu, \tilde{A}_\nu]^2 + \bar{c} \, [\, X_\mu, \, [\, X_\mu, \, c\,]\,] - \tilde{\bar{\psi}} \, \Gamma_\mu \, [\, X_\mu, \, \tilde{\psi}\,] \Bigg) \,, \end{aligned} \tag{B.7}$$

$$\begin{aligned} S_{\text{int}} = N \ \mathrm{tr} \Bigg(& -[\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu] - \frac{1}{4} \, [\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}_\mu, \tilde{A}_\nu] \, + \\ & + \frac{2}{3} \, i \, \alpha \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, \tilde{A}_i \, \tilde{A}_j \, \tilde{A}_k + \bar{c} \, [\, X_\mu, \, [\, \tilde{A}_\mu, \, c\,]\,] - \tilde{\bar{\psi}} \, \Gamma_\mu \, [\, \tilde{A}_\mu, \, \tilde{\psi}\,] \Bigg) \, . \end{aligned} \tag{B.8}$$

The linear terms in $\tilde{A}_\mu$ cancel since X_μ satisfies the classical equation of motion.

Noting that the background configuration X_μ includes a factor of α, we can rescale the fluctuations as $\tilde{A}_\mu \mapsto \alpha \, \tilde{A}_\mu$, $c \mapsto \alpha \, c$, $\bar{c} \mapsto \alpha \, \bar{c}$, $\tilde{\psi} \mapsto \alpha^{\frac{3}{2}} \, \tilde{\psi}$, $\tilde{\bar{\psi}} \mapsto \alpha^{\frac{3}{2}} \, \tilde{\bar{\psi}}$ so that all the terms in the total action S_{total} become proportional to α^4. This means that the expansion parameter of the present perturbation theory is $\frac{1}{\alpha^4}$.

The free energy W is defined by

$$\mathrm{e}^{-W} = \int \mathrm{d}\tilde{A} \, \mathrm{d}c \, \mathrm{d}\bar{c} \, \mathrm{d}\tilde{\psi} \, \mathrm{d}\tilde{\bar{\psi}} \, \mathrm{e}^{-S_{\text{total}}} \,, \tag{B.9}$$

which can be calculated as a perturbative expansion $W = \sum_{j=0}^{\infty} W_j$, where $W_j = \mathrm{O}(\alpha^{4(1-j)})$. The classical part is simply given by $W_0 = S_{\mathrm{b}}[X]$. In order to evaluate the one-loop term W_1, we note that the kinetic terms can be written as

$$S_{\rm kin} = N\ {\rm tr}\left(\frac{1}{2}\,\tilde{A}_\nu(\mathcal{P}_\lambda)^2\tilde{A}_\nu + \bar{c}\,(\mathcal{P}_\lambda)^2 c\right) - N\ {\rm tr}\left(\tilde{\bar{\psi}}\Gamma_\mu\mathcal{P}_\mu\tilde{\psi}\right), \tag{B.10}$$

where we have introduced the operator $\mathcal{P}_\mu$ which acts on a traceless $N\times N$ matrix M as

$$\mathcal{P}_\mu M \equiv [X_\mu, M]\,. \tag{B.11}$$

Then the one-loop term can be expressed as

$$W_1 = W_{1,\rm b} + W_{1,\rm f}\,, \tag{B.12}$$
$$W_{1,\rm b} = \mathcal{T}r\ \log\left\{N\,(\mathcal{P}_\mu)^2\right\}\,, \tag{B.13}$$
$$W_{1,\rm f} = -\mathcal{T}r'\ \log\left(\,N\,\Gamma_\mu\,\mathcal{P}_\mu\,\right), \tag{B.14}$$

where the symbol $\mathcal{T}r$ denotes the trace in the (N^2-1)-dimensional linear space which consists of traceless $N\times N$ matrices, and $\mathcal{T}r'$ includes the trace over spinor indices as well.

B.1 Single fuzzy sphere

Let us first consider the single fuzzy sphere $X_\mu = A_\mu^{(\rm S^2)}$. The classical part is given by

$$W_0 = -\frac{1}{24}\,N^2\,\alpha^4\,(N^2-1)\,, \tag{B.15}$$

and the one-loop terms can be written as

$$W_{1,\rm b} = \mathcal{T}r\ \log\left(\,N\,\alpha^2\,\mathcal{Q}\,\right), \tag{B.16}$$
$$W_{1,\rm f} = -\mathcal{T}r'\ \log\left(\,N\,\alpha\,\mathcal{D}\,\right). \tag{B.17}$$

The operators $\mathcal{Q}$ and $\mathcal{D}$ are defined as

$$\mathcal{Q} = \sum_{i=1}^{3}(\mathcal{L}_i)^2\,, \qquad \mathcal{D} = \sum_{i=1}^{3}\sigma_i\,\mathcal{L}_i\,, \tag{B.18}$$

where $\mathcal{L}_i$ acts on a traceless $N\times N$ matrix M as $\mathcal{L}_i\,M \equiv [L_i^{(N)}, M]$. In order to evaluate the one-loop terms, we need to solve the eigenvalue problem of the operators $\mathcal{Q}$ and $\mathcal{D}$.

The eigenvectors of the operator $\mathcal{Q}$ are given by the "matrix spherical harmonics" Y_{lm} ($l = 0, 1, \cdots, N-1$ and $m = -l, \cdots, l$), which span a complete basis of the space of $N\times N$ matrices and have the properties analogous to the usual spherical harmonics such as

$$\frac{1}{N}\ {\rm tr}\left(Y_{lm}^\dagger Y_{l'm'}\right) = \delta_{l\,l'}\delta_{m\,m'}\,, \tag{B.19}$$
$$Y_{lm}^\dagger = (-1)^m Y_{l,-m}\,. \tag{B.20}$$

The corresponding eigenvalues are given by $l\,(l+1)$; i.e.,

$$\mathcal{Q}\,Y_{lm} = l\,(l+1)\,Y_{lm}\,. \tag{B.21}$$

Thus the one-loop term from the bosonic contribution is obtained as

$$W_{1,\mathrm{b}} = \sum_{l=1}^{N-1} (2\,l+1) \log \left[\, N\,\alpha^2\, l\,(l+1)\,\right] . \tag{B.22}$$

Here $l = 0$ has been omitted from the sum since the trace $\mathcal{T}r$ in (B.13) should be taken in the space of *traceless* $N \times N$ matrices.

In order to solve the eigenvalue problem of the operator $\mathcal{D}$, we note that

$$\mathcal{D} = \sum_{i=1}^{3} (\mathcal{J}_i)^2 - \mathcal{Q} - \frac{3}{4}\,, \tag{B.23}$$

where we have defined the "total angular momentum" operator

$$\mathcal{J}_i = \mathcal{L}_i + \frac{\sigma_i}{2}\,. \tag{B.24}$$

By making a linear combination of eigenvectors of $\mathcal{Q}$ with the eigenvalue $l\,(l+1)$, we can construct the eigenvectors of both $\sum_{i=1}^{3} (\mathcal{J}_i)^2$ and $\mathcal{J}_3$ with the eigenvalues $j\,(j+1)$ and m, respectively, where j can be either $j = l+\frac{1}{2}$ $(l = 0, \cdots, N-1)$ or $j = l-\frac{1}{2}$ $(l = 1, \cdots, N-1)$, and m takes half-integer values in the range $|m| \le j$. Explicitly, the eigenvectors are given by the "matrix spinorial-spherical harmonics"

$$\mathcal{Y}_{l+\frac{1}{2},m} = \sqrt{\frac{l+\frac{1}{2}+m}{2\,l+1}}\,Y_{l,m-\frac{1}{2}} \otimes |\!\uparrow\rangle + \sqrt{\frac{l+\frac{1}{2}-m}{2\,l+1}}\,Y_{l,m+\frac{1}{2}} \otimes |\!\downarrow\rangle\,, \tag{B.25}$$

$$\mathcal{Y}'_{l-\frac{1}{2},m} = \sqrt{\frac{l+\frac{1}{2}-m}{2\,l+1}}\,Y_{l,m-\frac{1}{2}} \otimes |\!\uparrow\rangle - \sqrt{\frac{l+\frac{1}{2}+m}{2\,l+1}}\,Y_{l,m+\frac{1}{2}} \otimes |\!\downarrow\rangle\,, \tag{B.26}$$

for the cases $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$, respectively. Here the symbol $|\!\uparrow\rangle$ and $|\!\downarrow\rangle$ denotes the two-dimensional eigenvectors of σ_3 corresponding to the eigenvalues 1 and -1, respectively. From eq. (B.23), the "matrix spinorial-spherical harmonics" are also eigenvectors of $\mathcal{D}$ with the eigenvalues

$$\mathcal{D} = j\,(j+1) - l\,(l+1) - \frac{3}{4} \tag{B.27}$$

$$= \begin{cases} l & \text{for} \quad j = l + \dfrac{1}{2} \\ -(l+1) & \text{for} \quad j = l - \dfrac{1}{2}\,. \end{cases} \tag{B.28}$$

Namely we have the relation

$$\mathcal{D}\,\mathcal{Y}_{l+\frac{1}{2},m} = l\,\mathcal{Y}_{l+\frac{1}{2},m}\,, \tag{B.29}$$

$$\mathcal{D}\,\mathcal{Y}'_{l-\frac{1}{2},m} = -(l+1)\,\mathcal{Y}'_{l-\frac{1}{2},m}\,. \tag{B.30}$$

Thus the one-loop term from the fermionic contribution is obtained as

$$W_{1,\mathrm{f}} = -\left[\sum_{l=1}^{N-1} 2\,(l+1)\,\log\,(\,N\,\alpha\,l\,) + \sum_{l=1}^{N-1} 2\,l\,\log\{\,N\,\alpha\,(l+1)\,\}\right], \tag{B.31}$$

where $l = 0$ has been omitted from the first sum since the trace $\mathcal{T}r'$ in (B.14) should be taken in the space of *traceless* $N \times N$ matrices (and over the spinor indices). Let us rewrite the above expression as

$$W_{1,\mathrm{f}} = -W_{1,\mathrm{b}} - (N^2 - 1) \log N + \log N \,. \tag{B.32}$$

Thus the fermionic contribution cancel the bosonic contribution up to the α-independent constant. From (B.15) and (B.32) the one-loop free energy for the single fuzzy sphere is obtained at large N as

$$W_{1-\mathrm{loop}} \simeq N^2 \left(-\frac{1}{24} \tilde{\alpha}^4 - \log N \right) . \tag{B.33}$$

B.2 k coincident fuzzy spheres

Next we consider the k coincident fuzzy spheres $X_\mu = A_\mu^{(k\mathrm{S}^2)}$. The classical part of the free energy is given by

$$W_0 = -\frac{1}{24} \tilde{\alpha}^4 (n^2 - 1) \,. \tag{B.34}$$

In order to calculate the one-loop term, we consider the $n \times n$ version of the matrix spherical harmonics $Y_{lm}^{(n)}$, and define

$$Y_{lm}^{(a,b)} \equiv Y_{lm}^{(n)} \otimes \mathbf{e}^{(a,b)} \,, \tag{B.35}$$

where $\mathbf{e}^{(a,b)}$ denotes a $k \times k$ matrix whose (a, b) element is 1 and all the other elements are zero. The $N \times N$ matrices $Y_{lm}^{(a,b)}$ form a complete basis of the space of $N \times N$ matrices, and they are the eigenvectors of the operator $(\mathcal{P}_\mu)^2$ for the present background; i.e.,

$$(\mathcal{P}_\mu)^2 \, Y_{lm}^{(a,b)} = \alpha^2 \, l \, (l + 1) \, Y_{lm}^{(a,b)} \,. \tag{B.36}$$

Let us note that the operator $(\mathcal{P}_\mu)^2$ has k^2 zero modes corresponding to $l = m = 0$ with arbitrary (a, b). The mode $\sum_{a=1}^{k} Y_{00}^{(a,a)}$ corresponds to the trace mode, which should be omitted due to the traceless condition. Here we omit the other zero modes by hand, and simply consider the non-zero modes. Then the one-loop term $W_{1,\mathrm{b}}$ from the bosonic contribution is obtained as

$$W_{1,\mathrm{b}} = k^2 \sum_{l=1}^{n-1} (2 \, l + 1) \, \log \left[\, N \, \alpha^2 \, l \, (l + 1) \, \right] . \tag{B.37}$$

The calculation of the fermionic one-loop term proceeds in the same way except that we have to use the matrix spinorial-spherical harmonics for each of k^2 blocks. In this case we have $2k^2$ zero modes, and two of them correspond to the trace mode. We omit the other zero modes by hand and obtain

$$W_{1,\mathrm{f}} = -k^2 \left[\sum_{l=1}^{n-1} 2 \, (l + 1) \log \left(\, N \, \alpha \, l \, \right) + \sum_{l=1}^{n-1} 2 \, l \, \log \left\{ \, N \, \alpha \, (l + 1) \, \right\} \right] . \tag{B.38}$$

The cancellation between the bosonic contribution and the fermionic one occurs here as well, and the one-loop free energy is given by

$$W_{1-\mathrm{loop}} \simeq N^2 \left(-\frac{1}{24 \, k^2} \tilde{\alpha}^4 - \log N \right) . \tag{B.39}$$

C. One-loop calculation of various observables

In this section we apply the perturbation theory discussed in the previous section to the one-loop calculation of various observables which are studied by Monte Carlo simulations in this paper. We take the background to be k coincident fuzzy spheres $X_\mu = A_\mu^{(k\mathrm{S}^2)}$, but the results for the single fuzzy sphere can be readily obtained by setting $k = 1$. As in appendix B.2, we omit the zero modes for $k \geq 2$.

We note that the number of loops in the relevant diagrams can be less than the order of $1/\alpha^4$ in the perturbative expansion since we are expanding the theory around a nontrivial background. At the one-loop level, the only nontrivial task is to evaluate the tadpole $\langle(\tilde{A}_\mu)_{ij}\rangle$ explicitly, which, however, turns out to vanish due to supersymmetry.

C.1 Propagators and the tadpole

The propagators for $\tilde{A}_\mu$, the ghosts and the fermion fields are given respectively as

$$\left\langle (\tilde{A}_\mu)_{ij} (\tilde{A}_\nu)_{kl} \right\rangle_0 = \delta_{\mu\nu} \frac{1}{n} \sum_{ab} \sum_{l=1}^{n-1} \sum_{m=-l}^{l} \frac{1}{N\,\alpha^2\, l\,(l+1)} \left(Y_{lm}^{(a,b)}\right)_{ij} \left(Y_{lm}^{(a,b)\dagger}\right)_{kl} , \tag{C.1}$$

$$\left\langle (c)_{km} (\bar{c})_{pq} \right\rangle_0 = \frac{1}{n} \sum_{ab} \sum_{l=1}^{n-1} \sum_{m=-l}^{l} \frac{1}{N\,\alpha^2\, l\,(l+1)} \left(Y_{lm}^{(a,b)}\right)_{ij} \left(Y_{lm}^{(a,b)\dagger}\right)_{kl} , \tag{C.2}$$

$$\begin{aligned}\left\langle (\psi)_{ij} (\bar{\psi})_{kl} \right\rangle_0 = & -\frac{1}{n} \sum_{ab} \sum_{l=0}^{n-1} \sum_{m=-l-\frac{1}{2}}^{l+\frac{1}{2}} \frac{1}{N\,\alpha\, l} \left(\mathcal{Y}_{l+\frac{1}{2},m}^{(a,b)}\right)_{ij} \left(\mathcal{Y}_{l+\frac{1}{2},m}^{(a,b)\dagger}\right)_{kl} + \\ & + \frac{1}{n} \sum_{ab} \sum_{l=1}^{n-1} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \frac{1}{N\,\alpha\,(l+1)} \left(\mathcal{Y}'^{(a,b)}_{l-\frac{1}{2},m}\right)_{ij} \left(\mathcal{Y}'^{(a,b)\dagger}_{l-\frac{1}{2},m}\right)_{kl} ,\end{aligned} \tag{C.3}$$

where the symbol $\langle\ \cdot\ \rangle_0$ refers to the expectation value calculated using the kinetic term S_{kin} only. The tadpole $\langle\tilde{A}_i\rangle_{1-\mathrm{loop}}$ $(i = 1, 2, 3)$ at the one-loop level can be calculated as

$$\begin{aligned}\langle\tilde{A}_i\rangle_{1-\mathrm{loop}} = & \left\langle N\tilde{A}_i \ \mathrm{tr}\ \left([\tilde{A}_\nu, \tilde{A}_\rho][X_\nu, \tilde{A}_\rho]\right)\right\rangle_0 - \left\langle N\tilde{A}_i\ \mathrm{tr}\ \left(\bar{c}\,[X_\nu, [\tilde{A}_\nu, c]]\right)\right\rangle_0 - \\ & - \left\langle N\tilde{A}_i\ \mathrm{tr}\ \left(\tilde{\bar{\psi}}\Gamma_\nu[\tilde{A}_\nu, \tilde{\psi}]\right)\right\rangle_0 .\end{aligned} \tag{C.4}$$

By redoing the calculation in ref. [35] in the present model, we find that the bosonic contribution and the fermionic contribution cancel each other even at finite N. We also find that $\langle\tilde{A}_4\rangle = 0$ to all orders in perturbation theory due to parity invariance $A_4 \mapsto -A_4$.

C.2 One-loop results for various observables

Using the propagator and the tadpole obtained in the previous section, we can evaluate various observables easily at the one-loop level. For instance the two-point function $\left\langle \frac{1}{N}\ \mathrm{tr}\,(A_\mu)^2\right\rangle$ can be evaluated as follows. Let us decompose it as

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(A_\mu)^2 \right\rangle = \left\langle \frac{1}{N} \sum_{i=1}^{3} \mathrm{tr}\,(A_i)^2 \right\rangle + \left\langle \frac{1}{N}\ \mathrm{tr}\,(A_4)^2 \right\rangle . \tag{C.5}$$

Each term on the r.h.s. can be calculated as

$$\left\langle \frac{1}{N}\sum_{i=1}^{3} \mathrm{tr}\,(A_i)^2 \right\rangle_{1-\mathrm{loop}} = \frac{1}{N}\sum_{i=1}^{3}\left[\mathrm{tr}\,(X_iX_i) + 2\ \mathrm{tr}\,\left(X_i\langle\tilde{A}_i\rangle_{1-\mathrm{loop}}\right) + \langle \mathrm{tr}\,(\tilde{A}_i)^2\rangle_0\right]$$
$$= \alpha^2\left[\frac{1}{4}(n^2-1) + 0 + \frac{3}{\alpha^4\,n^2}\sum_{l=1}^{n-1}\frac{2\,l+1}{l\,(l+1)}\right], \tag{C.6}$$

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(A_4)^2 \right\rangle_{1-\mathrm{loop}} = \left\langle \frac{1}{N}\ \mathrm{tr}\,(\tilde{A}_4)^2 \right\rangle_0 = \frac{1}{\alpha^2\,n^2}\sum_{l=1}^{n-1}\frac{2\,l+1}{l\,(l+1)}\,. \tag{C.7}$$

At large N with fixed $\tilde{\alpha} = \alpha\sqrt{N}$, we get

$$\frac{1}{N}\left\langle \frac{1}{N}\ \mathrm{tr}\,(A_\mu)^2 \right\rangle_{1-\mathrm{loop}} \simeq \frac{1}{N}\left\langle \frac{1}{N}\sum_{i=1}^{3} \mathrm{tr}\,(A_i)^2 \right\rangle_{1-\mathrm{loop}} \simeq \frac{1}{4\,k^2}\,\tilde{\alpha}^2\,. \tag{C.8}$$

The Chern-Simons term $\langle M\rangle$ can be evaluated as

$$\langle M\rangle_{1-\mathrm{loop}} = \frac{2\,i}{3\,N}\,\epsilon_{ijk}\left[\mathrm{tr}\,(X_iX_jX_k) + 3\ \mathrm{tr}\,\left(X_iX_j\langle\tilde{A}_k\rangle_{1-\mathrm{loop}}\right)\right]$$
$$= -\frac{1}{6}\,\alpha^3\,(n^2-1)\,. \tag{C.9}$$

At large N with fixed $\tilde{\alpha} = \alpha\sqrt{N}$, we obtain

$$\frac{1}{\sqrt{N}}\,\langle M\rangle_{1-\mathrm{loop}} \simeq -\frac{1}{6\,k^2}\,\tilde{\alpha}^3\,. \tag{C.10}$$

The observable $\langle \frac{1}{N}\ \mathrm{tr}\ F^2\rangle$ can be calculated in a similar manner, but we can also obtain it from the exact result (3.3) using (C.10) as

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(F_{\mu\nu})^2 \right\rangle_{1-\mathrm{loop}} = 6\left(1-\frac{1}{N^2}\right) - 3\,\alpha\,\langle M\rangle_{1-\mathrm{loop}} \tag{C.11}$$
$$\simeq \frac{1}{2\,k^2}\,\tilde{\alpha}^4 + 6\,. \tag{C.12}$$

C.3 Alternative derivation

Since tr F^2 and M are the operators that appear in the action S, we can obtain their expectation values easily by using the free energy calculated for the k coincident fuzzy sphere in appendix B. Let us deform the bosonic action as

$$S_{\mathrm{b}}(\beta_1,\beta_2,\alpha) = N^2\left[\frac{1}{4}\beta_1\ \mathrm{tr}\,(F_{\mu\nu})^2 + \beta_2\,\alpha\,M\right] \tag{C.13}$$

with two free parameters β_1, β_2, and define the corresponding free energy as

$$\mathrm{e}^{-W(\beta_1,\beta_2,\alpha)} = \int \mathrm{d}A\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{e}^{-S_{\mathrm{b}}(\beta_1,\beta_2,\alpha)-S_{\mathrm{f}}}\,. \tag{C.14}$$

Then $\langle \mathrm{tr}\,(F_{\mu\nu})^2 \rangle$ and $\langle M \rangle$ can be obtained by

$$\left\langle \frac{1}{N}\ \mathrm{tr}\,(F_{\mu\nu})^2 \right\rangle = \frac{4}{N^2}\ \frac{\partial W}{\partial \beta_1}\bigg|_{\beta_1=\beta_2=1}, \tag{C.15}$$

$$\langle M \rangle = \frac{1}{\alpha\, N^2}\ \frac{\partial W}{\partial \beta_2}\bigg|_{\beta_1=\beta_2=1}. \tag{C.16}$$

By rescaling the integration variables as $A_\mu \mapsto \beta_1^{-\frac{1}{4}}\, A_\mu$, $\psi \mapsto \beta_1^{\frac{1}{8}}\, \psi$ and $\bar{\psi} \mapsto \beta_1^{\frac{1}{8}}\, \bar{\psi}$, we get

$$W(\beta_1, \beta_2, \alpha) = \frac{3}{2}\,(N^2-1)\,\log \beta_1 + W\left(1, 1, \alpha\, \beta_1^{-3/4}\, \beta_2\right). \tag{C.17}$$

Using the one-loop result

$$W(1,1,\alpha)_{1-\mathrm{loop}} = -\frac{N^2}{24}\,\alpha^4\,(n^2-1) - k^2\,(n^2-1)\,\log N + k^2\,\log n\,, \tag{C.18}$$

we can reproduce (C.9) and (C.11).

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